Stoytcho S. Yazadjiev
Department of Theoretical Physics, Faculty of Physics, Sofia University,
5 James Bourchier Boulevard, 1164 Sofia, Bulgaria
E-mail: yazad@phys.uni-sofia.bg

We present a method for generating solutions in some scalar-tensor theories with a minimally coupled massless scalar field or irrotational stiff perfect fluid as a source. The method is based on the group of symmetries of the dilaton-matter sector in the Einstein frame. In the case of Barker's theory the dilaton-matter sector possesses $SU(2)$ group of symmetries. In the case of Brans-Dicke and the theory with "conformal coupling", the dilaton- matter sector has $SL(2,R)$ as a group of symmetries. We describe an explicit algorithm for generating exact scalar-tensor solutions from solutions of Einstein-minimally-coupled-scalar-field equations by employing the nonlinear action of the symmetry group of the dilaton-matter sector. In the general case, when the dilaton-matter sector may not possess nontrivial symmetries we can still generate exact scalar-tensor solutions form solutions of Einstein-minimally-coupled-scalar-field using the geodesics of the Riemmanian metric associated with the Einstein frame dialton-matter sector. As an illustration of the general techniques, examples of explicit exact solutions are constructed. A generalization of the method for scalar-tensor-Maxwell gravity is outlined.

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I. INTRODUCTION

Scalar-tensor theories of gravity are considered as the most natural generalizations of general relativity [1]- [4]. In these theories gravity is mediated not only by the metric of space-time but also by a scalar field (the so called gravitational scalar). Scalar-tensor theories contain arbitrary functions of the scalar field that determine the gravitational "constant" as a dynamical variable. From a theoretical point of view it should be noted that specific scalar-tensor theories arise naturally as a low-energy limit of string theory.

In the weak field limit scalar-tensor theories differ slightly from general relativity. In the strong field regime, however, the predictions of scalar-tensor theories may differ drastically from those of general relativity as it was shown in [5], [6].

Scalar-tensor theories have also attracted much interest in cosmology [7]- [18].

The progress in the understanding of scalar-tensor theories of gravity is closely connected with finding and investigation of exact solutions. A theoretical discussions of many aspects of the early universe, gravitational waves, gravitational collapse and the structure of the compact objects within the framework of scalar-tensor theories (as in general relativity) necessitates the use of exact solutions. Besides the theoretical motivation for construction of exact solutions, there is also a more practical one. With the advent of numerical calculations, exact solutions are useful as comparisons for numerical and approximate solutions and as checks of the computer codes.

Solving of scalar-tensor theories equations in the presence of a source is a difficult task due to their complexity in the general case. In fact, scalar-tensor gravity equations are much more complicated than Einstein equations. In the so called Einstein frame the sourceless scalar-tensor equations are reduced to Einstein equations with a minimally coupled scalar field. In this case much progress has been achieved in finding exact homogeneous and inhomogeneous cosomological solutions [19], [28]. Little has been done in solving scalar-tensor equations with a source. The known solutions are homogeneous cosmological solutions with a perfect fluid depending on the time coordinate only [10]- [12].

The purpose of this paper is to present a method for generating exact solutions to the gravity equations with a minimally coupled massless scalar field (MCSF) and irrotational stiff perfect fluid within the framework of some scalar-tensor theories whose Einstein frame dilaton-matter sector has nontrivial symmetries. In the general case, when the Einstein frame dilaton-matter sector may not possess nontrivial symmetries we also present a solution generating technique which allows us to construct exact scalar-tensor solutions starting with the solutions of Einstein-minimally-coupled-scalar-field (EMCSF).

The motivations to consider a MCSF as a source are the following. In view of the complexity of the equations of the scalar-tensor gravity, it is natural as a first step to consider a simple source. On the other hand, in different contexts, the scalar field (different from the gravitational scalar) plays an important role in modern physics: the scalar field has been proposed as a candidate for gravitational lensing [29], [30] and for dark matter at galaxies scales [31], as well as at cosmological scales [32], [33], [34].

Examples of explicit exact solutions are also considered.

II. GENERAL EQUATIONS

Scalar-tensor theories are described by the following action in the Jordan frame:

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-\tilde{g}}\left(F(\Phi)\tilde{R} - Z(\Phi)\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2U(\Phi)\right) + S_m\left[\Psi_m;\tilde{g}_{\mu\nu}\right]. \quad (1)$$

Here, G_* is the bare gravitational constant, $\tilde{R}$ is the Ricci scalar curvature with respect to the space-time metric $\tilde{g}_{\mu\nu}$. The dynamics of the scalar field Φ depends on the functions $F(\Phi)$, $Z(\Phi)$ and $U(\Phi)$. In order for the gravitons to carry positive energy the function $F(\Phi)$ must be positive. The action of matter depends on the material fields Ψ_m and the space-time metric $\tilde{g}_{\mu\nu}$ but does not involve the scalar field Φ in order for the weak equivalence principle to be satisfied.

Varying the action (1) we obtain the scalar-tensor field equations

$$F(\Phi)\left(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}\right) = 8\pi G_*\tilde{T}_{\mu\nu} + Z(\Phi)\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi\right) + \tilde{\nabla}_\mu\tilde{\nabla}_\nu F(\Phi) - \tilde{g}_{\mu\nu}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha F(\Phi) - \tilde{g}_{\mu\nu}U(\Phi),$$

$$2\bar{\omega}(\Phi)\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\Phi = 8\pi G_*\frac{dF(\Phi)}{d\Phi}\tilde{T} - \frac{d\bar{\omega}}{d\Phi}\tilde{g}^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - 4U\frac{dF(\Phi)}{d\Phi} + 2\frac{dU(\Phi)}{d\Phi}F(\Phi),$$

$$\tilde{\nabla}_\mu\tilde{T}^\mu_\nu = 0$$

where $\tilde{\nabla}_\mu$ is the covariant derivative with respect to $\tilde{g}_{\mu\nu}$, $2\bar{\omega}(\Phi) = 2Z(\Phi)F(\Phi) + 3\left(\frac{dF(\Phi)}{d\Phi}\right)^2$ and $\tilde{T}^{\mu\nu} = (2/\sqrt{-g})\delta S_m/\delta\tilde{g}_{\mu\nu}$, $\tilde{T} = \tilde{T}^{\mu\nu}\tilde{g}_{\mu\nu}$.

The Jordan frame field equations (2) are rather complicated. It is much clearer to analyze the equations in the so-called Einstein frame. Let us introduce the new variables $g_{\mu\nu}$ and φ, and define

$$g_{\mu\nu} = F(\Phi)\tilde{g}_{\mu\nu} \ ,$$
$$\left(\frac{d\varphi}{d\Phi}\right)^2 = \frac{3}{4}\left(\frac{d\ln(F(\Phi))}{d\Phi}\right)^2 + \frac{Z(\Phi)}{2F(\Phi)} \quad (2)$$
$$A(\varphi) = F^{-1/2}(\Phi) \ ,$$
$$2V(\varphi) = U(\Phi)F^{-2}(\Phi) \ .$$

From now on we will refer to $g_{\mu\nu}$ and φ as Einstein frame metric and dilaton field correspondingly.

In the Einstein frame the action (1) takes the form

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 4V(\varphi)\right) + S_m[\Psi_m; A^2(\varphi)g_{\mu\nu}] \quad (3)$$

where R is the Ricci scalar curvature with respect to the Einstein frame metric $g_{\mu\nu}$. The Einstein frame field equations then are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_*T_{\mu\nu} + 2\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - 2V(\varphi)g_{\mu\nu} \ ,$$

$$\nabla^\mu\nabla_\mu\varphi = -4\pi G_*\alpha(\varphi)T + \frac{dV(\varphi)}{d\varphi} \ , \quad (4)$$

$$\nabla_\mu T^\mu_\nu = \alpha(\varphi)T\partial_\nu\varphi \ .$$

Here $\alpha(\varphi) = \frac{d\ln(A(\varphi))}{d\varphi}$ and the Einstein frame energy-momentum tensor is $T_{\mu\nu} = A^2(\varphi)\tilde{T}_{\mu\nu}$.

III. SCALAR-TENSOR THEORIES WITH A MINIMALLY COUPLED SCALAR FIELD AS A SOURCE AND SYMMETRIES OF DILATON - MATTER SECTOR

We consider scalar-tensor theories with a minimally coupled (massless) scalar filed (MCSF) σ as a matter source. The Jordan frame action for the scalar field is

$$S_m = \frac{1}{16\pi G_*}\int d^4x\sqrt{-\tilde{g}}\left(-2\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right) \quad (5)$$

In what follows we will consider only the special form of the potential $U(\Phi) = 2\Lambda F^2(\Phi)$ or $V(\varphi) = \Lambda$.

The full Einstein frame action then is

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g}\left(R - \Lambda + \mathcal{L}_{DM}\right) \quad (6)$$

where

$$\mathcal{L}_{DM} = -2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 2A^2(\varphi)g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \quad (7)$$

is the dilaton-matter sector of the theory Lagrangian.

Remarkably, in the case of some specific scalar-tensor theories the dilaton matter-sector (7) of the theory possesses hidden symmetries which allow us to generate exact solutions. In order to unveil these symmetries we define a two-dimensional abstract Riemanian space with a metric

$$dL^2 = d\varphi^2 + A^2(\varphi)d\sigma^2. \quad (8)$$

Solution generation techniques consist in finding invariant transformations of the dilaton-matter sector of the Lagrangian in the action (6). This is equivalent to finding the isometry group of the metric (8).

A. Barker's theory

Barker's theory [35] is described by the functions $F(\Phi) = \Phi$ and $Z(\Phi) = (4-3\Phi)/\Phi(2\Phi-2)$ corresponding to $A^2(\varphi) = \cos^2(\varphi)$. The metric (8) then is

$$dL^2 = d\varphi^2 + \cos^2(\varphi) d\sigma^2 \quad . \tag{9}$$

This metric can be considered as the standard metric on the unit 2-sphere. Indeed, taking into account that the theory is invariant under the transformation $\sigma \longrightarrow \sigma + const$ we may restrict the values of σ in an appropriate interval.

It is more convenient to present the metric in the well known complex form. To do so we introduce the complex variable

$$z = \cot(\varphi/2 + \pi/4) \, e^{i\sigma} \quad . \tag{10}$$

We obtain then

$$dL^2 = 4 \frac{dz d\bar{z}}{(1 + z\bar{z})^2} \quad . \tag{11}$$

The metric is invariant under the transformations

$$z \longrightarrow \frac{az + b}{-\bar{b}z + \bar{a}} \tag{12}$$

where $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$.

The symmetries of the dilaton-matter sector can be used to generate new solutions from known ones. In the case of Barker's theory, any solution of Einstein equations with a minimally coupled scalar field (EMCSF) and cosmological term will be a solution of the Einstein frame scalar-tensor-MCSF equations with cosmological term and with vanishing φ or σ as it can be seen from the field equations (4). Therefore, we can construct Barker counterparts to any solution of the EMCSF equations using the nonlinear action of the group $SU(2)$.

Taking as a seed solution $z_0 = e^{i\sigma_0}$ and performing a $SU(2)$ - transformation with the matrix

$$\begin{pmatrix} \cos(\beta/2) e^{i(\delta+\gamma)/2} & \sin(\beta/2) e^{i(\delta-\gamma)/2} \\ -\sin(\beta/2) e^{-i(\delta-\gamma)/2} & \cos(\beta/2) e^{-i(\delta+\gamma)/2} \end{pmatrix} \tag{13}$$

where $0 \leq \beta, \gamma, \delta < 2\pi$ we obtain a new solution z which in terms of φ and σ reads

$$\varphi = \arcsin\left[\sin(\beta)\cos(\sigma_0 + \gamma)\right] \quad , \tag{14}$$

$$\sigma = \arcsin\left[\frac{cos(\delta)\sin(\sigma_0 + \gamma) + \cos(\beta)\sin(\delta)\cos(\sigma_0 + \gamma)}{\sqrt{1 - \sin^2(\beta)\cos^2(\sigma_0 + \gamma)}}\right] \quad . \tag{15}$$

Once having the solution in the Einstein frame it is easy to recover the corresponding Jordan frame one. The results we summarize in the following proposition

Proposition: *Let $(g_{\mu\nu}, \sigma_0)$ be a solution to the EMCSF equations with a cosmological term Λ. Then $(\tilde{g}_{\mu\nu}, \Phi, \sigma)$ form a solution to the Barker-MCSF equations with a cosmological potential $U(\Phi) = 2\Lambda\Phi^2$ where σ is given by (15) and*

$$\tilde{g}_{\mu\nu} = \left(1 - \sin^2(\beta)\cos^2(\sigma_0 + \gamma)\right) g_{\mu\nu} \quad , \tag{16}$$

$$\Phi = \frac{1}{1 - \sin^2(\beta)\cos^2(\sigma_0 + \gamma)} \quad . \tag{17}$$

B. Theory with "conformal coupling"

The theory with "conformal coupling" (TCC) is characterized by the functions $F(\Phi) = 1 - \frac{1}{6}\Phi^2$ and $Z(\Phi) = 1$ which correspond to $A^2(\varphi) = \cosh^2(\varphi/\sqrt{3})$. The metric (8) then takes the form

$$dL^2_{TCC} = d\varphi^2 + \cosh^2(\varphi/\sqrt{3}) d\sigma^2 \quad . \tag{18}$$

This is a metric on the pseudo-sphere PS^2. The group of isometries of the metric is $SL(2,R)$. Let us introduce the complex variable

$$\zeta = e^{\sigma/\sqrt{3}} \tanh(\varphi/\sqrt{3}) + i \frac{e^{\sigma/\sqrt{3}}}{\cosh(\varphi/\sqrt{3})}. \tag{19}$$

The metric then can be written in the Klein form

$$dL^2 = -12 \frac{d\zeta d\bar{\zeta}}{(\zeta - \bar{\zeta})^2}. \tag{20}$$

The group $SL(2,R)$ acts as follows:

$$\zeta \longrightarrow \frac{A\zeta + B}{C\zeta + D} \tag{21}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,R)$.

Any solution to the EMCSF equations is also a solution to the equations of the theory with "conformal coupling" with a vanishing σ or φ. As a seed solution we take $(g_{\mu\nu}, \sigma_0)$ (i.e. with $\varphi_0 = 0$). Applying a $SL(2,R)$ transformation we obtain a three-parametric solution of the Einstein frame scalar-tensor equations with MCSF and a cosmological term:

$$A^2(\varphi) = 1 + \left(ACe^{\sigma_0/\sqrt{3}} + BDe^{-\sigma_0/\sqrt{3}}\right)^2 \quad , \tag{22}$$

$$e^{2\sigma/\sqrt{3}} = \frac{1 + \left(ACe^{\sigma_0/\sqrt{3}} + BDe^{-\sigma_0/\sqrt{3}}\right)^2}{\left(C^2 e^{\sigma_0/\sqrt{3}} + D^2 e^{-\sigma_0/\sqrt{3}}\right)^2} \quad . \tag{23}$$

We can summarize the results in the following

Proposition: *Let* $(g_{\mu\nu},\sigma_0)$ *be a solution to the EMCSF equations with a cosmological term* Λ. *Then* $(\tilde{g}_{\mu\nu},\Phi,\sigma)$ *form a solution to the TCC-MCSF equations with a cosmological potential* $U(\Phi) = 2\Lambda\left(1-\frac{1}{6}\Phi^2\right)^2$ *where* σ *is given by (23) and*

$$\tilde{g}_{\mu\nu} = \left(1+\left(ACe^{\sigma_0/\sqrt{3}}+BDe^{-\sigma_0/\sqrt{3}}\right)^2\right)g_{\mu\nu} \ ,$$
$$\frac{1}{6}\Phi^2 = \frac{\left(ACe^{\sigma_0/\sqrt{3}}+BDe^{-\sigma_0/\sqrt{3}}\right)^2}{1+\left(ACe^{\sigma_0/\sqrt{3}}+BDe^{-\sigma_0/\sqrt{3}}\right)^2} \ . \tag{24}$$

C. Brans-Dicke theory

Brans-Dicke theory is characterized by the functions $F(\Phi) = \Phi$ and $Z(\Phi) = \omega/\Phi$ which correspond to $A^2(\varphi) = e^{2\alpha\varphi}$ where $\alpha^2 = 1/(3+2\omega)$ *. In the Brans-Dicke case the metric (8) takes the form

$$dL^2_{BD} = d\varphi^2 + e^{2\alpha\varphi}d\sigma^2. \tag{25}$$

The metric (25), up to the parameter α, also appears in the so called dilaton-axion gravity [36]. It should be noted, however, that we consider (25) in a different physical context. The symmetries of dilaton-axion sector are well known, but for completeness we will describe them again in our context.

Introducing the complex variable

$$\xi = \sigma + \frac{i}{\alpha}e^{-\alpha\varphi} \tag{26}$$

we can write the metric (25) in the Klein form

$$dL^2_{BD} = -\frac{4}{\alpha^2}\frac{d\xi d\bar{\xi}}{(\xi-\bar{\xi})^2}. \tag{27}$$

This metric is equivalent, up to a constant conformal factor, to the metric (18). The dilaton-matter sector is $SL(2,R)$ invariant:

$$\xi \longrightarrow \frac{A\xi+B}{C\xi+D} \tag{28}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,R)$.

Let us consider a solution $(g_{\mu\nu},\sigma_0)$ to the EMCSF equations. This solution is also a solution to the Einstein frame scalar- tensor equations with $\xi_0 = \frac{i}{\alpha}e^{-\alpha\sigma_0}$. The $SL(2,R)$ transformation with the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

gives a new solution $z = \sigma + \frac{i}{\alpha}e^{-\alpha\varphi}$ where

$$e^{\alpha\varphi} = D^2e^{\alpha\sigma_0} + \frac{1}{\alpha^2}C^2e^{-\alpha\sigma_0}$$
$$\sigma = \frac{\frac{1}{\alpha^2}ACe^{-\alpha\sigma_0}+BDe^{\alpha\sigma_0}}{D^2e^{\alpha\sigma_0}+\frac{1}{\alpha^2}C^2e^{-\alpha\sigma_0}} \ .$$

In order to write these formulas in more convenient form we redefine the parameters B and C as follows:

$$B \longrightarrow \frac{B}{\alpha}, \quad C \longrightarrow \alpha C \ .$$

After the performed redefinition we obtain

$$e^{\alpha\varphi} = D^2e^{\alpha\sigma_0} + C^2e^{-\alpha\sigma_0} \ , \tag{29}$$

$$\sigma = \frac{1}{\alpha}\frac{ACe^{-\alpha\sigma_0}+BDe^{\alpha\sigma_0}}{D^2e^{\alpha\sigma_0}+C^2e^{-\alpha\sigma_0}} \ . \tag{30}$$

Since the metrics (18) and (25) are simply related by a constant conformal factor the above solution can be obtained from the corresponding one in the TCC. Indeed, taking the TCC-solution given by (22) and (23) and performing the transformations

$$\begin{aligned} e^{\sigma/\sqrt{3}}\tanh(\varphi/\sqrt{3}) &\longrightarrow \sigma \\ e^{-\sigma/\sqrt{3}}\cosh(\varphi/\sqrt{3}) &\longrightarrow \alpha e^{\alpha\varphi} \\ \sigma_0 &\longrightarrow \sqrt{3}\alpha\sigma_0 \end{aligned} \tag{31}$$

together with the redefinitions $A \to A/\sqrt{\alpha}$, $B \to B/\sqrt{\alpha}$, $C \to \sqrt{\alpha}C$, $D \to \sqrt{\alpha}D$ we obtain the Brans-Dicke solution given by (29) and (30).

Having once the solution in the Einstein frame we can recover the corresponding Brans-Dicke solution in Jordan frame:

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \left(D^2e^{\alpha\sigma_0}+C^2e^{-\alpha\sigma_0}\right)^2 g_{\mu\nu} \\ \Phi &= \left(D^2e^{\alpha\sigma_0}+C^2e^{-\alpha\sigma_0}\right)^{-2} \end{aligned} \tag{32}$$

In this way we proved the following proposition

Proposition: *Let* $(g_{\mu\nu},\sigma_0)$ *be a solution to the EMCSF equations with a cosmological term* Λ. *Then* $(\tilde{g}_{\mu\nu},\Phi,\sigma)$ *given by (30) and (32), form a solution to the Brans-Dicke equations with a MCSF and a cosmological potential* $U(\Phi) = 2\Lambda\Phi^2$.

D. Other theories

When the metric (8) has nontrivial isometries we can use them to generate new solutions from known ones. In the general case, however, the metric (8) has only trivial symmetry $\sigma \longrightarrow \sigma + constant$. That is why the solution generating procedure considered above fails. Nevertheless when the dilaton-matter sector does not possess nontrivial symmetries we can still generate scalar-tensor solutions starting with solutions to the EMCSF.

*We consider the case $\omega > -3/2$.

For this purpose we consider the geodesics of the metric (8). It is not difficult to see that if $(\varphi(\tau), \sigma(\tau))$ is an affinely parameterized geodesic and $(g_{\mu\nu}, \sigma_0)$ is a solution to the EMCSF equations with a cosmological term Λ, then $(g_{\mu\nu}, \varphi(\sigma_0), \sigma(\sigma_0))$ is a solution to the scalar-tensor-MCSF equations with a cosmological term Λ.

The geodesic equations for the metric (8) can be immediately written and we obtain

$$\int \frac{d\varphi}{\sqrt{1 - C^2/A^2(\varphi)}} = C_1 \pm \tau \ ,$$
$$\sigma = C_2 + C \int \frac{d\tau}{A^2(\varphi(\tau))} \tag{33}$$

where C, C_1 and C_2 are constants.

As a concrete example we consider the theory with $F(\Phi) = \Phi$ and $Z(\Phi) = \frac{1}{2}\frac{\Phi^2 - 3\Phi + 3}{\Phi(\Phi - 1)}$ corresponding to $A^2(\varphi) = 1/(1 + \varphi^2)$. Solving the geodesic equations for the metric $dL^2 = d\varphi^2 + \frac{d\sigma^2}{1+\varphi^2}$ we obtain

$$\varphi = \frac{1}{C}\sqrt{1 - C^2}\sin(C_1 + C\sigma_0) \ , \tag{34}$$

$$\sigma = C_2 + \frac{1 + C^2}{2C}\sigma_0 - \frac{1 - C^2}{4C^2}\sin(2C_1 + 2C\sigma_0) \tag{35}$$

where $0 < C^2 < 1$.

Therefore, if $(g_{\mu\nu}, \sigma_0)$ is a solution to the EMCSF equations with a cosmological term Λ, then $(\tilde{g}_{\mu\nu}, \Phi, \sigma)$ is a solution to the equations of $A^2(\varphi) = 1/(1 + \varphi^2)$-theory with a MCSF and a cosmological potential $U(\Phi) = 2\Lambda\Phi^2$ where σ is given by (35) and

$$\tilde{g}_{\mu\nu} = \frac{1}{\Phi} g_{\mu\nu} \ ,$$
$$\Phi = 1 + \frac{1 - C^2}{C^2}\sin^2(C_1 + C\sigma_0) \ . \tag{36}$$

IV. PERFECT FLUID SCALAR-TENSOR EQUATIONS

In the case of perfect fluid the Jordan frame energy-momentum tensor is

$$\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{p})\tilde{u}_\mu\tilde{u}_\nu + \tilde{p}\tilde{g}_{\mu\nu} \ . \tag{37}$$

Here $\tilde{u}_\mu$ is the fluid 4-velocity and $\tilde{\rho}$ and $\tilde{p}$ are the energy density and pressure respectively.

The corresponding Einstein energy-momentum tensor is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \ . \tag{38}$$

where the Einstein frame fluid 4-velocity, energy density and pressure are given by $u_\mu = A^{-1}(\varphi)\tilde{u}_\mu$, $\rho = A^4(\varphi)\tilde{\rho}$ and $p = A^4(\varphi)\tilde{p}$.

In the present work we consider only an irrotational perfect fluid. The 4-velocity then can be written as [27], [28]

$$\tilde{u}_\mu = \partial_\mu\sigma / \sqrt{-\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma} \tag{39}$$

where σ is the "velocity scalar potential".

Our further simplification is to consider only a stiff perfect fluid with equation of state $\tilde{\rho} = \tilde{p}$. In this case the fluid energy density and pressure can be written in terms of the potential σ as follows [27], [28]

$$8\pi G_*\tilde{\rho} = 8\pi G_*\tilde{p} = -\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma. \tag{40}$$

The fluid energy-momentum density then reads

$$8\pi G_*\tilde{T}_{\mu\nu} = 2\partial_\mu\sigma\partial_\nu\sigma - \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma \tag{41}$$

in the Jordan frame and

$$8\pi G_* T_{\mu\nu} = A^2(\varphi)\left(2\partial_\mu\sigma\partial_\nu\sigma - g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma\right) \tag{42}$$

in the Einstein frame.

As it seen the energy-momentum of an irrotational stiff perfect fluid coincides with the one for a massless scalar field with the restriction $\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma < 0$. Therefore the dilaton-matter sector will possess the same symmetries as in the case of massless scalar field considered in the previous section. Without going into detail we will formulate the propositions from the previous section in terms of a perfect fluid.

A. Barker's theory

Proposition: *Let $(g_{\mu\nu}, \rho_0, u^{(0)}_\mu)$ be a solution to the Einstein equations with an irrotational stiff perfect fluid and a cosmological term Λ. Then $(\tilde{g}_{\mu\nu}, \Phi, \tilde{\rho}, \tilde{u}_\mu)$ is a solution to the Barker's equations with an irrotational stiff perfect fluid and cosmological potential $U(\Phi) = 2\Lambda\Phi^2$ if*

$$\tilde{g}_{\mu\nu} = \left[1 - \sin^2(\beta)\cos^2(\sigma_0 + \gamma)\right] g_{\mu\nu} \ ,$$
$$\Phi = \frac{1}{1 - \sin^2(\beta)\cos^2(\sigma_0 + \gamma)} \ ,$$
$$\tilde{\rho} = \frac{\cos^2(\beta)}{\left[1 - \sin^2(\beta)\cos^2(\sigma_0 + \gamma)\right]^3}\ \rho_0 \ , \tag{43}$$
$$\tilde{u}_\mu = u^{(0)}_\mu\sqrt{1 - \sin^2(\beta)\cos^2(\sigma_0 + \gamma)} \ ,$$
$$8\pi G_*\rho_0 = -g^{\mu\nu}\partial_\mu\sigma_0\partial_\nu\sigma_0 \ .$$

B. Theory with "conformal coupling"

Proposition: *Let $(g_{\mu\nu}, \rho_0, u^{(0)}_\mu)$ be a solution to the Einstein equations with a irrotational stiff perfect fluid and a cosmological term Λ. Then $(\tilde{g}_{\mu\nu}, \Phi, \tilde{\rho}, \tilde{u}_\mu)$ is* *a solution to the TCC equations with an irrotational stiff perfect fluid and a cosmological potential* $U(\Phi) = 2\Lambda\left(1-\frac{1}{6}\Phi^2\right)^2$ *if*

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= \left(1+\left(ACe^{\sigma_0/\sqrt{3}}+BDe^{-\sigma_0/\sqrt{3}}\right)^2\right) g_{\mu\nu} \ , \\
\frac{1}{6}\Phi^2 &= \frac{\left(ACe^{\sigma_0/\sqrt{3}}+BDe^{-\sigma_0/\sqrt{3}}\right)^2}{1+\left(ACe^{\sigma_0/\sqrt{3}}+BDe^{-\sigma_0/\sqrt{3}}\right)^2} \ , \\
\tilde{\rho} &= \frac{(AD+BC)^2}{\left(1+\left(ACe^{\sigma_0/\sqrt{3}}+BDe^{-\sigma_0/\sqrt{3}}\right)^2\right)^3}\rho_0 \ , \qquad (44) \\
\tilde{u}_\mu &= u^{(0)}_\mu \sqrt{1+\left(ACe^{\sigma_0/\sqrt{3}}+BDe^{-\sigma_0/\sqrt{3}}\right)^2} \ , \\
8\pi G_*\rho_0 &= -g^{\mu\nu}\partial_\mu\sigma_0\partial_\nu\sigma_0 \ .
\end{aligned}$$

C. Brans-Dicke theory

Proposition: *Let* $(g_{\mu\nu}, \rho_0, u^{(0)}_\mu)$ *be a solution to the Einstein equations with an irrotational stiff perfect fluid and a cosmological term* Λ. *Then* $(\tilde{g}_{\mu\nu}, \Phi, \tilde{\rho}, \tilde{u}_\mu)$ *is a solution to the Brans-Dicke equations with an irrotational stiff perfect fluid and a cosmological potential* $U(\Phi) = 2\Lambda\Phi^2$ *if*

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= \left(D^2e^{\alpha\sigma_0}+C^2e^{-\alpha\sigma_0}\right)^2 g_{\mu\nu} \ , \\
\Phi &= \frac{1}{\left(D^2e^{\alpha\sigma_0}+C^2e^{-\alpha\sigma_0}\right)^2} \ , \\
\tilde{\rho} &= \frac{4C^2D^2}{\left(D^2e^{\alpha\sigma_0}+C^2e^{-\alpha\sigma_0}\right)^6}\rho_0 \ , \qquad (45) \\
\tilde{u}_\mu &= u^{(0)}_\mu\left(D^2e^{\alpha\sigma_0}+C^2e^{-\alpha\sigma_0}\right) \ , \\
8\pi G_*\rho_0 &= -g^{\mu\nu}\partial_\mu\sigma_0\partial_\nu\sigma_0 \ .
\end{aligned}$$

D. $A^2(\varphi) = 1/(1+\varphi^2)$-theory

Proposition: *Let* $(g_{\mu\nu}, \rho_0, u^{(0)}_\mu)$ *be a solution to the Einstein equations with an irrotational stiff perfect fluid and a cosmological term* Λ. *Then* $(\tilde{g}_{\mu\nu}, \Phi, \tilde{\rho}, \tilde{u}_\mu)$ *is a solution to the* $A^2(\varphi)$*-theory equations with an irrotational stiff perfect fluid and a cosmological potential* $U(\Phi) = 2\Lambda\Phi^2$ *if*

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= \frac{1}{\Phi}g_{\mu\nu} \ , \\
\Phi &= 1+\frac{1-C^2}{C^2}\sin^2(C_1+C\sigma_0) \ , \\
\tilde{\rho} &= C^2\left(1+\frac{1-C^2}{C^2}\sin^2(C_1+C\sigma_0)\right)^3\rho_0 \ , \qquad (46) \\
\tilde{u}_\mu &= \frac{u^{(0)}_\mu}{\sqrt{1+\frac{1-C^2}{C^2}\sin^2(C_1+C\sigma_0)}} \ , \\
8\pi G_*\rho_0 &= -g^{\mu\nu}\partial_\mu\sigma_0\partial_\nu\sigma_0 \ .
\end{aligned}$$

V. EXAMPLES OF EXACT SOLUTIONS

As an illustration of the general techniques we consider some concrete examples of exact solutions. In order not to fall into tedious calculations we shall consider only simple seed solutions and set $\Lambda = 0$.

A. Cosmological solutions

Our first examples are cosmological solutions with an irrotational stiff perfect fluid. Consider the metric

$$ds_0^2 = -dt^2 + t^{2/3}\left(dx^2+dy^2+dz^2\right) \ . \qquad (47)$$

This is a flat Friedman-Robertson-Walker metric for stiff perfect fluid. The fluid energy density is $8\pi G_*\rho_0 = 1/3t^2$ and the 4-velocity is $u_\mu = -\partial_\mu t$. The fluid velocity potential is $\sigma_0 = -1/\sqrt{3}\ln(t)$.

1. Barker's theory

Using the solution generating procedure for Barker's theory and putting $\gamma = 0$ we obtain

$$\begin{aligned}
d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu &= \left(1-sin^2(\beta)\cos^2(\frac{1}{\sqrt{3}}\ln(t))\right)ds_0^2 \ , \\
\Phi &= \frac{1}{1-\sin^2(\beta)\cos^2(\frac{1}{\sqrt{3}}\ln(t))} \ , \\
8\pi G_*\tilde{\rho} &= \frac{1}{3t^2}\frac{\cos^2(\beta)}{\left(1-sin^2(\beta)\cos^2(\frac{1}{\sqrt{3}}\ln(t))\right)^3} \\
\tilde{u}_\mu &= -\sqrt{1-sin^2(\beta)\cos^2(\frac{1}{\sqrt{3}}\ln(t))}\ (\partial_\mu t) \, . \qquad (48)
\end{aligned}$$

2. Theory with "conformal coupling"

In the case of the theory with "conformal coupling" the solution generating techniques give

$$\begin{aligned}
d\tilde{s}^2 &= \left(1+t^{-2/3}\left(AC+BDt^{2/3}\right)^2\right)ds_0^2 \ , \\
\frac{1}{6}\Phi^2 &= \frac{\left(AC+BDt^{2/3}\right)^2}{t^{2/3}+\left(AC+BDt^{2/3}\right)^2} \ ,
\end{aligned}$$

$$8\pi G_*\tilde{\rho} = \frac{1}{3t^2}\frac{(AD+BC)^2}{\left(1+t^{-2/3}\left(AC+BDt^{2/3}\right)^2\right)^3} \ , \qquad (49)$$
$$\tilde{u}_\mu = -\sqrt{1+t^{-2/3}\left(AC+BDt^{2/3}\right)^2}\ (\partial_\mu t)$$

3. $A^2(\varphi) = 1/(1+\varphi^2)$-theory

Solution generating formulas (46) give

$$d\tilde{s}^2 = \frac{1}{1+\frac{1-C^2}{C^2}\sin^2\left(\frac{C}{\sqrt{3}}\ln(t)\right)}\ ds_0^2 \ ,$$
$$\Phi = 1+\frac{1-C^2}{C^2}\sin^2\left(\frac{C}{\sqrt{3}}\ln(t)\right) \ , \qquad (50)$$
$$8\pi G_*\tilde{\rho} = \frac{C^2}{3t^2}\left[1+\frac{1-C^2}{C^2}\sin^2\left(\frac{C}{\sqrt{3}}\ln(t)\right)\right]^3 \ ,$$
$$\tilde{u}_\mu = \frac{-\partial_\mu t}{\sqrt{1+\frac{1-C^2}{C^2}\sin^2\left(\frac{C}{\sqrt{3}}\ln(t)\right)}} \ .$$

Here we have set $C_1 = 0$.

B. Static spherically symmetric solutions

Our next examples are static spherically symmetric solutions. As a seed solution we take the Janis-Newman-Winicour solution [37] to the EMCSF equations:

$$ds_0^2 = -f_0^{2\lambda}dt^2 + f_0^{-2\lambda}dr^2 + f_0^{2(1-\lambda)}r^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \ ,$$
$$\sigma_0 = \sqrt{1-\lambda^2}\ln(f_0) \qquad (51)$$

where $f_0^2 = 1-\frac{2M}{r}$ and $0\le\lambda\le 1$.

1. Barker's theory

The corresponding Barker's theory solution is

$$d\,\tilde{s}^2 = \left[1-\sin^2(\beta)\cos^2\left(\sqrt{1-\lambda^2}\ln(f_0)+\pi/2\right)\right]ds_0^2 \ ,$$
$$\Phi = \frac{1}{1-\sin^2(\beta)\cos^2\left(\sqrt{1-\lambda^2}\ln(f_0)+\pi/2\right)} \ , \qquad (52)$$
$$\sigma = -\arcsin\left[\frac{\cos(\beta)\cos\left(\sqrt{1-\lambda^2}\ln(f_0)+\pi/2\right)}{\sqrt{1-\sin^2(\beta)\cos^2\left(\sqrt{1-\lambda^2}\ln(f_0)+\pi/2\right)}}\right] .$$

In order to insure $\lim_{r\to\infty}\sigma = 0$ and $\lim_{r\to\infty}\Phi = 1$ we have restricted ourselves with the subgroup $SO(2)\subset SU(2)$.

2. Theory with "conformal coupling"

In the case of the theory with "conformal coupling" we obtain the following static solution:

$$d\tilde{s}^2 = \left[1+\left(ACf_0^{\sqrt{(1-\lambda^2)/3}}+BDf_0^{-\sqrt{(1-\lambda^2)/3}}\right)^2\right]ds_0 \ ,$$
$$\frac{1}{6}\Phi^2 = \frac{\left(ACf_0^{\sqrt{(1-\lambda^2)/3}}+BDf_0^{-\sqrt{(1-\lambda^2)/3}}\right)^2}{1+\left(ACf_0^{\sqrt{(1-\lambda^2)/3}}+BDf_0^{-\sqrt{(1-\lambda^2)/3}}\right)^2} \ , \qquad (53)$$
$$\sigma = \frac{\sqrt{3}}{2}\ln\left[\frac{1+\left(ACf_0^{\sqrt{(1-\lambda^2)/3}}+BDf_0^{-\sqrt{(1-\lambda^2)/3}}\right)^2}{C^2f_0^{\sqrt{(1-\lambda^2)/3}}+D^2f_0^{-\sqrt{(1-\lambda^2)/3}}}\right] .$$

Here, in order to insure $\lim_{r\to\infty}\sigma = 0$ and $\lim_{r\to\infty}\Phi = 0$ we should take $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(2)\subset SU(2)$.

3. $A^2(\varphi) = 1/(1+\varphi^2)$-theory

Using solution generating formulas (35), (36) and putting $C_1 = 0$ and $C_2 = 0$ we obtain

$$d\tilde{s}^2 = \frac{1}{1+\frac{1-C^2}{C^2}\sin^2(C\sqrt{1-\lambda^2}\ln(f_0))}\ ds_0^2 \ ,$$
$$\Phi = 1+\frac{1-C^2}{C^2}\sin^2(C\sqrt{1-\lambda^2}\ln(f_0)) \ , \qquad (54)$$
$$\sigma = \frac{1+C^2}{2C}\sqrt{1-\lambda^2}\ln(f_0) - \frac{1-C^2}{4C^2}\sin(2C\sqrt{1-\lambda^2}\ln(f_0)) \ .$$

Of course, we may construct many more and much more complicated exact scalar-tensor solutions taking as seed solutions the known solutions of EMCSF equations in the literature (for example see ([21]), ([26])). The physical properties of every class generated scalar-tensor solutions, however, require a separate investigation.

VI. CONCLUSION

In this paper we have presented a method for generating exact solutions in some scalar-tensor theories of gravity with a MCSF or irrotational stiff perfect fluid and with a potential term of a special kind for the gravitational scalar. The method is based on the symmetries of the dilaton-matter sector in the Einstein frame. In the case of Barker's theory the dilaton-matter sector possess a group of symmetry $SU(2)$. In the case of Brans-Dicke and the theory with "conformal coupling" the dilaton-matter sector is $SL(2,R)$-symmetric.

We have described an explicit algorithm for generating exact scalar-tensor solutions starting from solutions to the EMCSF equations with a cosmological term by employing the non-linear action of the symmetry group of the dilaton-matter sector.

In the general case, when the Riemmanian metric (8) associated with the dilaton-matter sector does not possess nontrivial symmetries we can still generate scalar-tensor solutions from solutions of EMCSF equations using the geodesics of the metric (8).

As an illustration of the solution generating techniques explicit exact solutions have also been constructed.

The solution generating method can be generalized for scalar-tensor-Maxwell gravity. Taking into account that the Maxwell action is conformally invariant we can immediately write the scalar-tensor-MCSF-Maxwell action in the Einstein frame

$$S = \frac{1}{16\pi G_*}\int d^4x\,(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \\ -2A^2(\varphi)g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - F_{\mu\nu}F^{\mu\nu} - \Lambda\big) \quad .$$

As is seen the dilaton-MCSF sector possesses the same symmetries (when they exist) as in the case of absence of Maxwell field and they can be employed to generate new solutions with electromagnetic field from known ones. In the general case, when the metric associated with the dilaton-MCSF sector may not possess nontrivial symmetries we can construct exact scalar-tensor solutions in presence of electromagnetic field using the geodesics of the dilaton-MCSF sector metric.

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